Exam D (Part II) **Name**

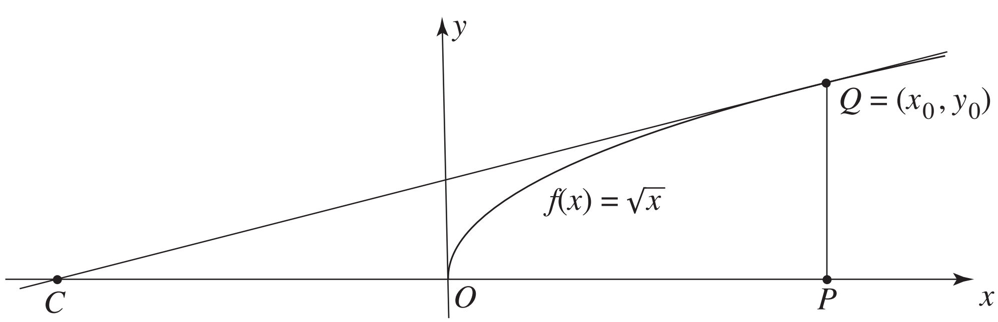
Please Note: Calculators may be used in elementary or trig mode only, not in calculus mode.

1. State the Fundamental Theorem of Calculus. Then discuss the question with accompanying diagrams: Does any continuous function (this is one that has a graph without gaps or breaks) have an antiderivative?

2. By setting up and evaluating a definite integral, derive an estimate for the sum

$$\sqrt{4}\cdot\tfrac{1}{10{,}000}+\sqrt{4+\tfrac{1}{10{,}000}}\cdot\tfrac{1}{10{,}000}+\sqrt{4+\tfrac{2}{10{,}000}}\cdot\tfrac{1}{10{,}000}+\cdots+\sqrt{7+\tfrac{9{,}999}{10{,}000}}\cdot\tfrac{1}{10{,}000}.$$

3. The graph of the function $f(x)=\sqrt{x}$ with $x\geq 0$ is shown below along with a random point



$Q(x_0,y_0)$ on the graph. The line through Q and C is tangent to the graph at Q. The point P has coordinates $P=(x_0,0)$.

a. Compute the area under the graph of $f(x) = \sqrt{x}$ and over the interval $0 \le x \le x_0$ in terms of x_0.

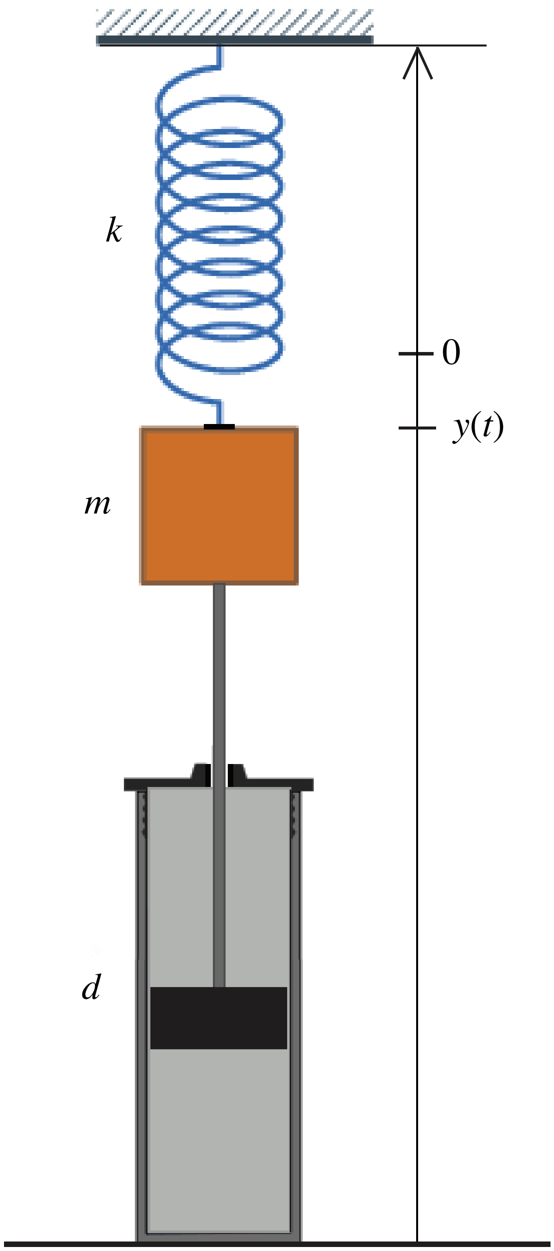
b. Compute the area of the triangle ΔCPQ. [You'll need to determine the equation of the tangent in terms of x_0 first.] How is the areas of the triangle related to the area you found in (3a?

4. Find a function $y = f(x)$ that satisfies the differential equation $y'' + 4y' + 12y = 0$ as well as the conditions $f(0) = 0$ and $f'(0) = 2\sqrt{2}$. Put the resulting $y = f(x)$ into the box below.

$f(x) =$

5. The figure below shows a system consisting of a rigidly attached spring (with spring constant k), a block of mass m, and a cylinder (filled with air or a fluid) that resists the motion proportionally to the velocity of the piston and hence the block (with damping constant d). See the figure below. The origin 0 of the vertical y-axis marks the endpoint of the spring at its natural length and $y(t)$ is the typical position of the top of the block.

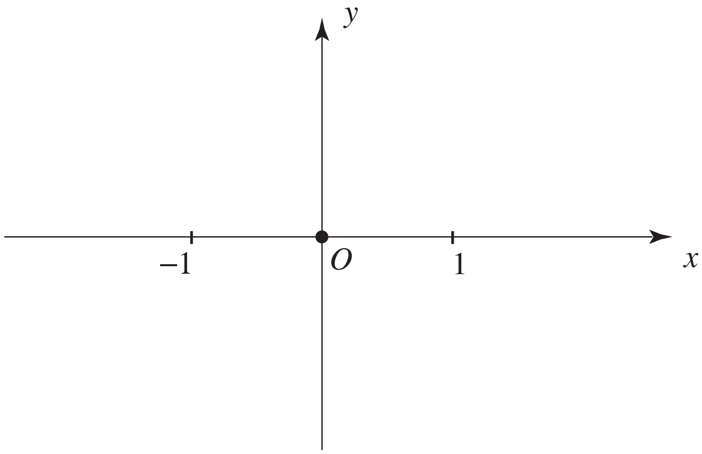
Write an expression for each of the forces involved and indicate its direction (in the situation that the diagram presents). Use your conclusions to derive a second order differential equation that the function $y(t)$ satisfies. Put your differential equation into the box below.



6. Consider the polar function $r = f(\theta) = \frac{1}{1-\sin\theta}$.

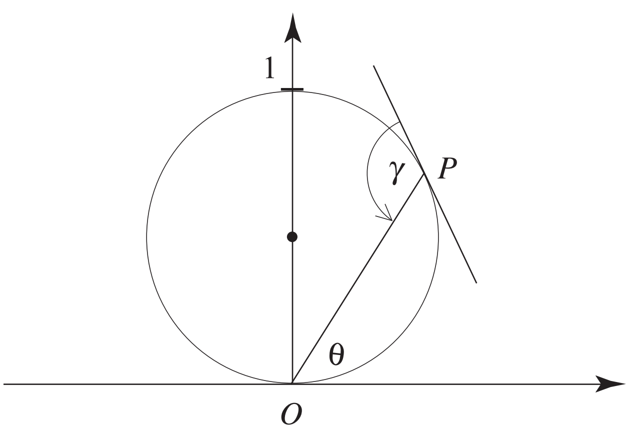
a. Show that $r = f(\theta)$ is always positive.

b. Convert the equation $r = \frac{1}{1-\sin\theta}$ into a Cartesian equation and simplify it. Identify its graph (it is a conic section) and sketch the graph into the coordinate plane below.



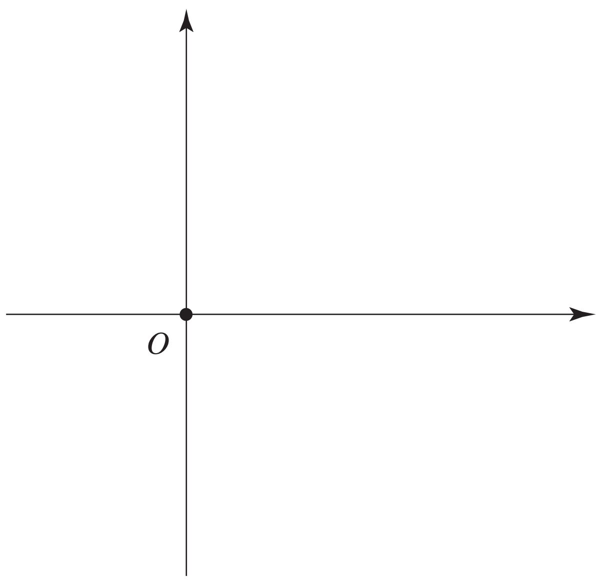
7. The graph of the polar function $r = f(\theta) = \sin\theta$ is the circle shown in the figure below. The center of the circle is shown. A typical point $P = (\theta, \sin\theta)$ on the circle along with the tangent to the circle is also shown. The angle between this tangent and the segment OP is denoted by γ.

a. Use the figure to verify that $\gamma = \pi - \theta$.



b. Use the identities $\sin(\frac{\pi}{2} - \theta) = \cos\theta$ and $\cos(\frac{\pi}{2} - \theta) = \sin\theta$ to verify the formula $f'(\theta) = f(\theta) \cdot \tan(\gamma - \frac{\pi}{2})$.

8. Consider the function $r = f(\theta) = \frac{12}{4\sin\theta + cos\theta}$. Sketch its graph into the plane below (the plane features a polar coordinate system and a superimposed xy-coordinate system).



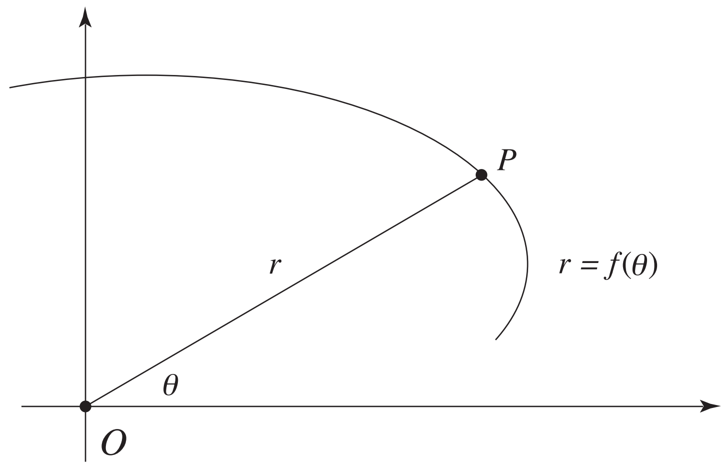
a. Use the graph to evaluate $\displaystyle\int_0^{\frac{\pi}{2}} \sqrt{f(\theta)^2 + f'(\theta)^2}\, d\theta$.

Answer:

b. Use the graph again to evaluate $\displaystyle\int_0^{\frac{\pi}{4}} \tfrac{1}{2}\, f(\theta)^2\, d\theta$.

Answer:

9. The figure below shows a particle P in motion subject only to a centripetal force that acts in the direction of the point O. It is known that P is never at O. A polar coordinate system is given and the trajectory of the particle is the graph of the differentiable polar function $r = f(\theta)$. The angle $\theta = \theta(t)$ and the distance $r(t) = f(\theta(t))$ from O to P are also differentiable functions of t. In the problems below you may make use of the fact that $r(t)^2\theta'(t)$ is equal to a constant c.



i. Verify that $2\frac{dr}{dt} \cdot \frac{d\theta}{dt} + r \cdot \frac{d^2\theta}{dt^2} = 0$. (This equation is an important step in the derivation of the polar force equation.)

ii. Consider P at two different times t_0 and t_1 in its orbit. Suppose that $t_0 \leq t_1$ and let A be the area that is swept out by the segment OP during the time interval $[t_0, t_1]$. Put this information into the diagram above and then verify that $A = \frac{1}{2}c(t_1 - t_0)$ where c is the constant mentioned earlier. (Notice that this is Kepler's second law).

Formulas and expressions:

$$\int u\,dv = uv - \int v\,du \qquad x = r\cos\theta \quad y = r\sin\theta \qquad Ay'' + By' + Cy = 0$$

$$y = D_1 e^{r_1 x} + D_2 e^{r_2 x} \qquad y = D_1 e^{2x} + D_2 x e^{2x} \qquad y = e^{ax}(D_1 \cos bx + D_2 \sin bx)$$

$$a = \tfrac{d}{1-\varepsilon^2} \qquad b = \tfrac{d}{\sqrt{1-\varepsilon^2}} \qquad a = \tfrac{d}{\varepsilon^2-1} \qquad b = \tfrac{d}{\sqrt{\varepsilon^2-1}} \qquad f'(\theta) = f(\theta)\cdot\tan(\gamma - \tfrac{\pi}{2})$$

$$f(x) = \tfrac{s}{d^2}x^2, \quad \tan\alpha = \tfrac{2s}{d}, \quad T(x) = w\sqrt{\left(\tfrac{d^2}{2s}\right)^2 + x^2}, \quad T_d = wd\sqrt{\left(\tfrac{d}{2s}\right)^2 + 1}, \quad T_0 = \tfrac{wd^2}{2s}.$$

$$ab\pi \qquad a^2 = b^2 + c^2 \qquad \tfrac{x^2}{a^2} + \tfrac{y^2}{b^2} = 1 \quad \kappa = \tfrac{ab\pi}{T} \qquad F = ma \quad f(t)\cdot r = I\cdot\alpha(t) \qquad \tfrac{d}{dx}\sin x = \cos x$$

$$\int_a^b \sqrt{1 + f'(x)^2}\,dx \qquad \int_a^b \pi f(x)^2\,dx \quad 2\pi\int_a^b f(x)\sqrt{1 + f'(x)^2}\,dx$$